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CRACK PROBLEM FOR A SHALLOW SHELL WITH A FLEXIBLE COATING

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In this paper, we study the effect of a flexible coating on the stress-strain state and limit equilibrium of a cracked thin shell. An analogous problem for a plate in tension with an isolated cut has been studied in [1] and that for plates with a system of cracks has been treated in [2, 3].

1. We consider an isotropic Kirchhoff–Love shell weakened by a rectilinear plane through crack oriented along the curvature line of the middle surface. Let one of the shell faces be covered by a flexible coating, which deforms jointly with the substrate and and can withstand considerable stresses. The sides of the crack are opened by self-equilibrated membrane forces; the rest of the boundary is free of external load. Let us formulate the problem of the influence of the coating on the equilibrium of a cracked shell.

We use the Cartesian system of coordinates $Oxyz$ (Fig. 1a). The stress-strain state of the shell outside of the crack is described by the equations of shallow shell theory [4]

$$\Delta\Delta\varphi - \frac{B}{R}\Delta_k w = 0, \quad \Delta\Delta w + \frac{1}{DR}\Delta_k \varphi = 0, \quad (x, y) \in \mathbf{R}^2 \backslash L. \tag{1.1}$$

Here, φ is the stress function; w is the deflection of the shell; $\Delta = \partial^2/\partial x^2 + \partial^2/\partial y^2$, $\Delta_k = \beta_2 \partial^2/\partial x^2 + \beta_1 \partial^2/\partial y^2$; $\beta_1 = R/R_1$; $\beta_2 = R/R_2$; $R = \min(|R_1|, |R_2|)$; R_1, and R_2 are the principal curvature radii of the normal sections of the middle surface; $B = 2Eh$; $D = 2Eh^3/(3(1-\nu^2))$; E and ν are the Young's modulus and Poisson's ratio of the material of the shell; h is the half-thickness of the shell; and L is a segment of the x axis along which a cut of length $2l$ is situated.

Membrane forces and bending moments are assumed to vanish at infinity

$$N_x = N_{xy} = N_y = 0, \quad M_y = M_{xy} = M_y = 0, \quad (x, y) \to \infty. \tag{1.2}$$

We consider the crack in the shell with the flexible coating as a cut whose sides are connected by means of hinges at one of the faces $z = sh$ ($s = +1$ or $s = -1$) (Fig. 1b). Following [1], we write the boundary conditions of the symmetrical problem in the form

$$[v] - sh[\vartheta_y] = 0, \quad x \in L; \tag{1.3}$$

$$N_y = -p + T, \quad M_y = shT, \quad x \in L, \tag{1.4}$$

where $[v]$ is the cut opening in the middle surface of the shell, $[\vartheta_y]$ is the jump of the rotation angle of the normal ($\vartheta_y = \partial w/\partial y$), $-p$ is the specified uniformly distributed load, and T is the reaction at the hinge.

Eliminating the unknown contact reaction from relation (1.4), we arrive at the static contact condition

$$M_y = sh(N_y + p), \quad x \in L. \tag{1.5}$$

Relations (1.1)–(1.3) and (1.5) are the boundary-value problem describing the elastic equilibrium of a shallow shell weakened by a cut with hinge-connected edges under the action of a symmetric load.

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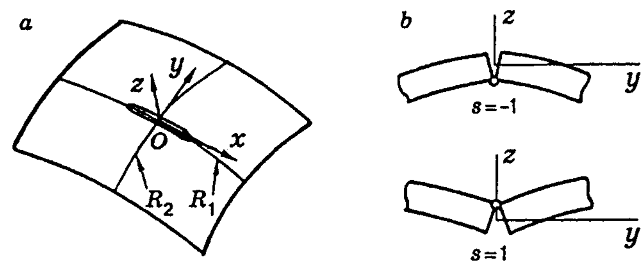

Fig. 1

2. Let us construct the integral equation of the formulated problem. We write the integral representation of the forces and moments at the line $y = 0$ in terms of derivatives of the jump functions

$$N_y(x,0) = \frac{B}{4\pi}\int_L \left\{K_{11}(\xi - x)[v]'(\xi) - K_{13}(\xi - x)a[\vartheta_y]'(\xi)\right\} d\xi,$$
$$M_y(x,0) = \frac{Ba}{4\pi}\int_L \left\{K_{31}(\xi - x)[v]'(\xi) - K_{33}(\xi - x)a[\vartheta_y]'(\xi)\right\} d\xi. \tag{2.1}$$

The kernels of these representations can be expressed in terms of the Fourier integrals [5, 6]:

$$K_{jk}(z) = [\delta_{jk}\mathrm{Re} + (1-\delta_{jk})\mathrm{Im}]\int_0^\infty g_{jk}(\gamma\sqrt{-i}/s)\sin zs ds, \quad j,k = 1,3. \tag{2.2}$$

Here,

$$g_{11}(\rho) = r(\rho)/\omega(\rho); \quad g_{13}(\rho) = g_{31}(\rho) = -r(\rho)[1 + \nu/\omega(\rho)];$$
$$g_{33}(\rho) = r(\rho)[2 - 2\nu + \beta_1\rho^2 + \omega(\rho) - \nu^2/\omega(\rho)]; \qquad r(\rho) = 2[2 + \beta_1\rho^2 + 2\omega(\rho)]^{-1/2};$$
$$\omega(\rho) = (1 + \beta_2\rho^2)^{1/2}; \quad \rho = \gamma\sqrt{-i}/s; \quad \gamma = 1/\sqrt{Ra}; \quad a = h/\sqrt{3(1-\nu^2)}; \quad z = \xi - x;$$

δ_{jk} is the Kronecker delta.

Substituting expressions (2.1) into the boundary condition (1.5) and eliminating the jump of the rotation angle by means of equality (1.3), we obtain a singular integral equation for the jump of displacements

$$\frac{B}{4\pi}\int_L \left\{K_{11}(\xi - x) - \frac{2s}{\sqrt{3(1-\nu^2)}}K_{13}(\xi - x) + +\frac{1}{3(1-\nu^2)}K_{33}(\xi - x)\right\}[v]'(\xi)d\xi = -p, \quad x \in L, \tag{2.3}$$

whose solution has to satisfy the additional condition

$$[v](\partial L) = 0. \tag{2.4}$$

If we introduce nondimensional coordinates $t = x/l$ and $\tau = \xi/l$, the problem (2.3) and (2.4) takes the form

$$\frac{B}{4\pi}\int_{-1}^{1} K(\tau - t)[v]'(\tau)d\tau = -p, \quad t \in (-1,1), \quad [v](\pm 1) = 0,$$
$$K(\zeta) = K_{11}(\zeta) - 2sK_{13}(\zeta)/\sqrt{3(1-\nu^2)} + K_{33}(\zeta)/(3(1-\nu^2)), \ \ K_{jk}(\zeta) = lK_{jk}(l\zeta), \ \ j,k = 1,3, \ \ \zeta = \tau - t. \tag{2.5}$$

3. Problem (2.5) was solved by the small parameter method in a first shell approximation. It is well known [5] that kernels (2.2) can be expanded in terms of the small parameter $\lambda = l\gamma = (l/\sqrt{Rh})(3(1-\nu^2))^{1/4}$

as follows:

$$K_{jk}(\zeta) = \frac{a_{jk0}}{\zeta} + \lambda \sum_{p=1}^{\infty} (a_{jkp} + b_{jkp} \ln \lambda|\zeta|)(\lambda\zeta)^p.$$

The zero and first coefficients of the expansion are given by the formulas

$$a_{110} = 1, \quad a_{130} = a_{310} = 0, \quad a_{330} = 3 - 2\nu - \nu^2,$$

$$a_{111} = -\frac{\beta_1 + 5\beta_2}{32}\pi \mathrm{B} - \frac{\sqrt{-\beta_1\beta_2}}{24}\frac{3\beta_1^2 - 22\beta_1\beta_2 + 15\beta_2^2}{(\beta_1 - \beta_2)^2}\eta(-\beta_1\beta_2),$$

$$a_{331} = \frac{(5 + 2\nu + \nu^2)\beta_1 + (1 + 2\nu + 5\nu^2)\beta_2}{32}\pi \mathrm{B}$$

$$+ \frac{\sqrt{-\beta_1\beta_2}}{24}\frac{3(5 + 2\nu + \nu^2)\beta_1^2 - 2(11 + 2\nu + 11\nu^2)\beta_1\beta_2 + 3(1 + 2\nu + 5\nu^2)\beta_2^2}{(\beta_1 - \beta_2)^2}\eta(-\beta_1\beta_2),$$

$$a_{131} = a_{311} = \left[\frac{3(1 + \nu)\beta_1^2 + 4(1 + 11\nu)\sqrt{\beta_1\beta_2^3} + (5 + 37\nu)\beta_2^2}{48(\sqrt{\beta_1} - \sqrt{\beta_2})^2} + b_{131}\left(\ln\frac{\gamma_0(\sqrt{\beta_1} + \sqrt{\beta_2})}{4} - 1\right)\right]\eta(\beta_1\beta_2)$$

$$+\left[\frac{(1 + \nu)(\beta_1 + 3\beta_2)}{16} + \frac{\beta_2^2}{12}\frac{3(1 - 3\nu)\beta_1 - (1 - 7\nu)\beta_2}{(\beta_1 - \beta_2)^2} + b_{131}\left(\ln\frac{\gamma_0\sqrt{|\beta_1 - \beta_2|}}{4} - 1\right)\right]\eta(-\beta_1\beta_2),$$

$$b_{111} = b_{331} = 0, \; b_{131} = b_{311} = \frac{(1 + \nu)\beta_1 + (1 + 5\nu)\beta_2}{8}, \qquad \mathrm{B} = 1 - \frac{2}{\pi}\arctan\frac{2\sqrt{-\beta_1\beta_2}}{\beta_1 + \beta_2}\eta(-\beta_1\beta_2)$$

[$\eta(\ldots)$ is the Heaviside function, $\ln \gamma_0 = 0.5772\ldots$ is the Euler constant].

For the corresponding coefficients of the representation

$$K(\zeta) = \frac{a_0}{\zeta} + \lambda \sum_{p=1}^{\infty} (a_p + b_p \ln \lambda|\zeta|)(\lambda\zeta)^p \tag{3.1}$$

we obtain the expressions

$$a_0 = a_{110} + a_{330}/(3(1 - \nu^2)), \quad a_1 = a_{111} - 2sa_{131}/\sqrt{3(1 - \nu^2)} + a_{331}/(3(1 - \nu^2)), \quad b_1 = -2sb_{131}/\sqrt{3(1 - \nu^2)}.$$

Bearing the expansion (3.1) in mind, we find a solution of the problem in a first shell approximation

$$[v](t) = \frac{pl}{B}\Phi(\lambda, t), \quad [\vartheta_y](t) = \frac{pshl}{3(1 - \nu^2)D}\Phi(\lambda, t). \tag{3.2}$$

Here

$$\Phi(\lambda, t) = \frac{4\sqrt{(1 - t^2)}}{a_0}\left\{1 - \frac{\lambda^2}{2a_0}\left[a_1 + b_1\left(\frac{2}{3} + \frac{t^2}{3} + \ln\frac{\lambda}{2}\right)\right] + O(\lambda^4 \ln^2 \lambda)\right\}.$$

Substituting (3.2) into formula (2.1), we have with the same degree of accuracy the distribution of the contact reaction at the hinge connection

$$T(t) = \frac{p(3 + \nu)}{2(3 + 2\nu)}\left\{1 - \frac{\lambda^2}{2a_0}\left[A_1 + B_1\left(\frac{1}{2} + t^2 + \ln\frac{\lambda}{2}\right)\right] + O(\lambda^4 \ln^2 \lambda)\right\},$$
$$A_1 = a_{111} + 2s\nu a_{131}/((3 + \nu)\sqrt{3(1 - \nu^2)}) - a_{331}/a_{330}, \quad B_1 = 2s\nu b_{131}/((3 + \nu)\sqrt{3(1 - \nu^2)}). \tag{3.3}$$

We calculate the force and moment intensity coefficient K_1 and K_3 [5] in the neighborhood of the cut ends from the formulas

$$K_1 = -\frac{1}{4}Ba_{110}\sqrt{l}\lim_{t\to 1}\sqrt{(1 - t^2)}[v]'(t) = \frac{3p\sqrt{l}(1 + \nu)}{2(3 + 2\nu)}F(\lambda),$$

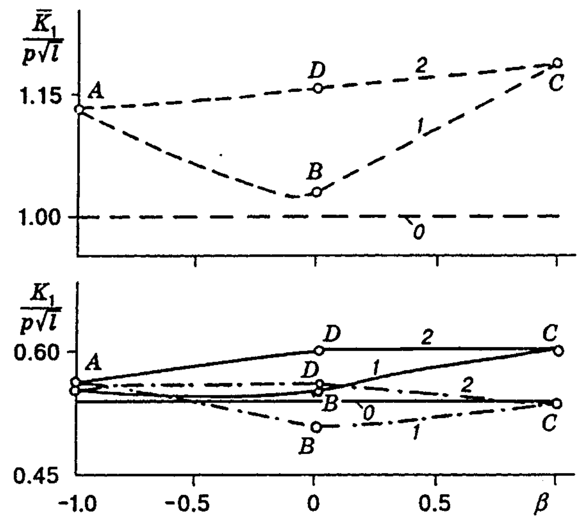

Fig. 2

$$K_3 = \frac{1}{4} D a_{330} \sqrt{l} \lim_{t\to 1} \sqrt{(1-t^2)} [\vartheta_y]'(t) = -\frac{psh\sqrt{l}(3+\nu)}{2(3+2\nu)} F(\lambda), \tag{3.4}$$

$$F(\lambda) = 1 - \frac{\lambda^2}{2a_0} \left[a_1 + b_1 \left(1 + \ln \frac{\lambda}{2} \right) \right] + O(\lambda^4 \ln^2 \lambda).$$

If $\lambda = 0$, the formulas (3.2)–(3.4) yield a solution to the problem for a cut with hinge-connected rims in a plate in tension [1].

We write the expressions of the function $F(\lambda)$ corresponding to the particular values of the parameters β_1 and β_2 that are most important in practice:

(a) for a pseudospherical shell with a cut along the curvature line ($\beta_1 = -\beta_2 = \pm 1$)

$$F(\lambda) = 1 + \frac{\lambda^2}{3-\nu-2\nu^2} \left[\frac{5-\nu-10\nu^2}{64} \pi \mp s\sqrt{3(1-\nu^2)} \left(\frac{1+11\nu}{48} + \frac{\nu}{4} \ln \frac{\gamma_0 \sqrt{2}\lambda}{8} \right) \right] + O(\lambda^4 \ln^2 \lambda); \tag{3.5}$$

(b) for a cylindrical shell with a cut along the ruler: ($\beta_1 = 1$ and $\beta_2 = 0$)

$$F(\lambda) = 1 - \frac{\lambda^2}{3-\nu-2\nu^2} \left[\frac{1+\nu+2\nu^2}{64} \pi - s\sqrt{3(1-\nu^2)} \left(\frac{1+\nu}{32} + \frac{1+\nu}{16} \ln \frac{\gamma_0 \lambda}{8} \right) \right] + O(\lambda^4 \ln^2 \lambda);$$

(c) for a cylindrical shell with a cut along the generator ($\beta_1 = 0$ and $\beta_2 = 1$)

$$F(\lambda) = 1 + \frac{\lambda^2}{3-\nu-2\nu^2} \left[\frac{7-\nu-10\nu^2}{64} \pi + s\sqrt{3(1-\nu^2)} \left(\frac{5+37\nu}{96} + \frac{1+5\nu}{16} \ln \frac{\gamma_0 \lambda}{8} \right) \right] + O(\lambda^4 \ln^2 \lambda);$$

(d) for a spherical shell with a meridional cut ($\beta_1 = \beta_2 = 1$)

$$F(\lambda) = 1 + \frac{\lambda^2}{3-\nu-2\nu^2} \left[\frac{3-\nu-6\nu^2}{32} \pi + s\sqrt{3(1-\nu^2)} \left(\frac{1+7\nu}{32} + \frac{1+3\nu}{16} \ln \frac{\gamma_0 \lambda}{4} \right) \right] + O(\lambda^4 \ln^2 \lambda).$$

Let us estimate the limit equilibrium of a shell in tension with a crack using the energy criterion of fracture under combined tension and bending [1, 7, 8]:

$$G = 2\gamma_*, \quad G = \frac{\pi}{4h^2 E} \left[K_1^2 + \frac{3(1+\nu)}{3+\nu} \left(\frac{K_3}{h} \right)^2 \right]. \tag{3.6}$$

Here, G is the energy flux to the crack tip and γ_* is the density of the effective surface energy of the material.

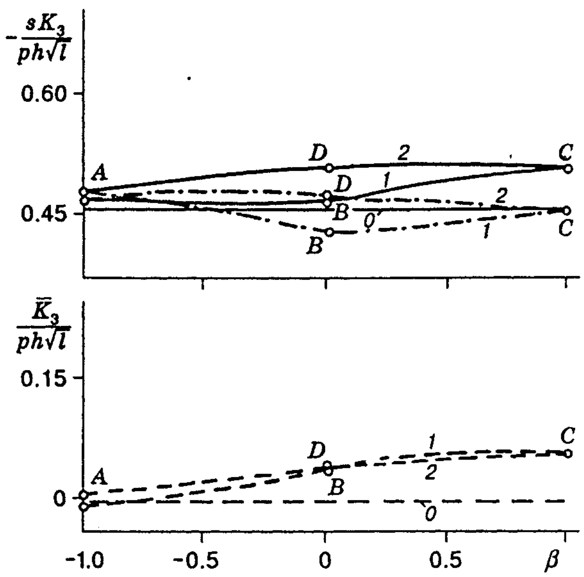

Fig. 3

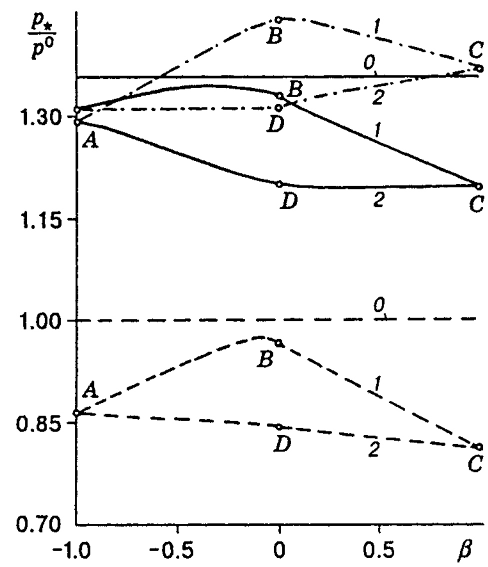

Fig. 4

Substituting relations (3.4) into the criterion, we find the the load leading to the crack propagation:

$$p_* = p^0\sqrt{\frac{2(3+2\nu)}{3(1+\nu)}}\left\{1+\frac{\lambda^2}{2a_0}\left[a_1+b_1\left(1+\ln\frac{\lambda}{2}\right)\right]+O(\lambda^4\ln^2\lambda)\right\} \tag{3.7}$$

($p^0 = \sqrt{8h^2E\gamma_*/(\pi l)}$ is the rupture extending force for a plate containing a crack with rims free of connections).

4. If we consider the problem of a cut in a shallow shell without a coating, after substituting the integral representations (2.1) into the boundary conditions $N_y = -p$ and $M_y = 0$, $x \in (-l, l)$ we obtain a system of integral equations whose solutions are the functions [5]

$$\begin{gathered}
[\bar{v}](t) = \frac{4pl\sqrt{(1-t^2)}}{Ba_{110}}\left[1-\frac{a_{111}}{2a_{110}}\lambda^2+O(\lambda^4\ln\lambda)\right], \\
[\bar{\vartheta}_y](t) = \frac{4phl\sqrt{(1-t^2)}}{Ba_{110}\sqrt{3(1-\nu^2)}}\frac{\lambda^2}{2a_{330}}\left[a_{311}+b_{311}\left(\frac{2}{3}+\frac{t^2}{3}+\ln\frac{\lambda}{2}\right)\right]+O(\lambda^4\ln^2\lambda).
\end{gathered} \tag{4.1}$$

To these jumps correspond the force and moment intensity coefficients

$$\begin{gathered}
\bar{K}_1 = p\sqrt{l}\left[1-\frac{a_{111}}{2a_{110}}\lambda^2+O(\lambda^4\ln\lambda)\right], \\
\bar{K}_3 = -\frac{ph\sqrt{l}}{\sqrt{3(1-\nu^2)}}\frac{\lambda^2}{2a_{110}}\left[a_{311}+b_{311}\left(1+\ln\frac{\lambda}{2}\right)\right]+O(\lambda^4\ln^2\lambda),
\end{gathered} \tag{4.2}$$

and also the critical load calculated with the same accuracy according to criterion (3.6)

$$\bar{p}_* = p^0\left[1+\frac{a_{111}}{2a_{110}}\lambda^2+O(\lambda^4\ln^2\lambda)\right]. \tag{4.3}$$

5. Let us discuss the results obtained. The graphs shown in Figs. 2 and 3 characterize the effect of the shell shape on the force and moment intensity coefficients for $\nu = 0.3$. Analogous relations for a nondimensional critical load are given in Fig. 4. Curves 0 correspond to the case of a plate in tension $\lambda = 0$ [1]. Curves 1 characterize crack orientation in the shell along the line of the maximum curvature ($\beta_1 = 1$ and $\beta_2 = \beta$)

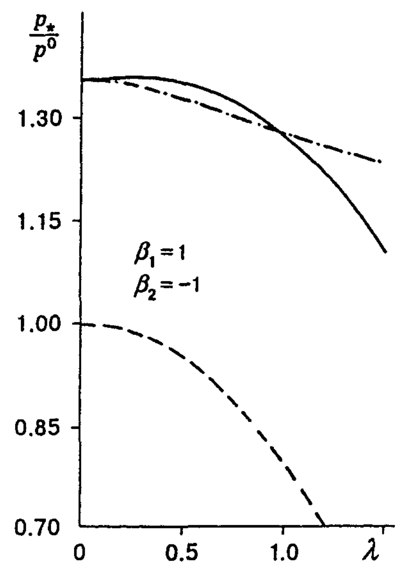

Fig. 5

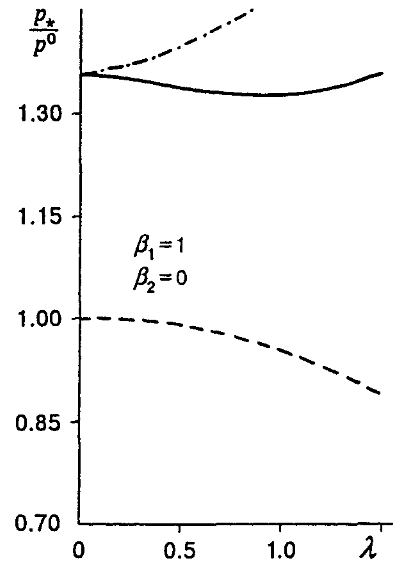

Fig. 6

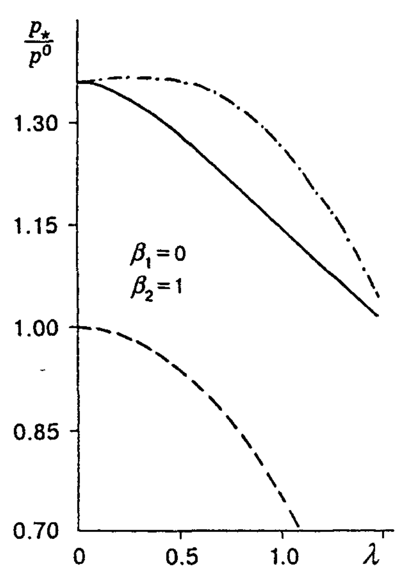

Fig. 7

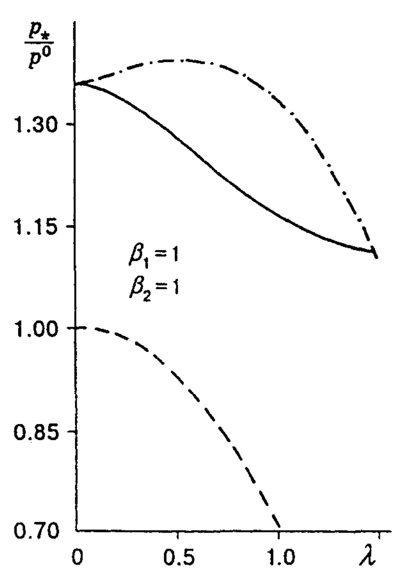

Fig. 8

and curves 2 characterize that along the line of the minimum curvature ($\beta_1 = \beta$ and $\beta_2 = 1$) for $\lambda = 0.8$. Here, points A and C correspond to pseudospherical and spherical shells, and points B and D correspond to a cylindrical shell with transverse and longitudinal cuts, respectively. Expressions (3.4) and (3.7) taking into account the presence of a flexible coating are represented by solid ($s = -1$) and dot-dashed ($s = 1$) curves. For comparison, the results (4.2) and (4.3) obtained in the classical formulation are shown by dashed lines.

As is seen from the above graphs, hinge connection leads to considerable reduction of the force intensity coefficients and to the appearance of considerable moment intensity coefficients. In the classical formulation the load-carrying capacity of a shell in tension with a crack is always less than that of a plate, whereas in the presence of a flexible coating the fracture load for a shell can be either greater or lesser than that for a plate. Indeed, the correction for curvature in formula (4.3) depends in a first approximation only on the expansion coefficients of the kernel $K_{11}(\zeta)$ and is negative for arbitrary values of β_1 and β_2. The multiplier $a_1 + b_1(1 + \ln(\lambda/2))$ appearing in expression (3.7), which takes into account the effect of coating, depends on the expansion coefficients of all kernels $K_{jk}(\zeta)$, and depending on the shell shape and the parameters s and λ, can be either positive or negative.

Figures 5–8 show more detailed depedences of the fracture load on the parameter λ, which are obtained for $\nu = 0.3$ for shells of the simplest geometry: a pseudospherical shell (Fig. 5), a cylindrical shell with transverse (Fig. 6) and longitudinal (Fig. 7) cuts, and a spherical shell (Fig. 8).

Since the crack orientation change for a pseudospherical shell is equivalent in a first approximation to the change of sign of the parameter s [see formula (3.5)], the plots corresponding to the case of $\beta_1 = -1$ and $\beta_2 = 1$ can be obtained by rearrangement of the solid and dashed curves in Fig. 5.

We note that a cut with hinge-connected rims in a shell in tension is characterized, in general, by a nonmonotonous dependence of the limiting load on the parameter λ.

The effect of a flexible coating on the stress-strain state and limiting equilibrium of a shell for larger values of the parameter λ as well as the range of applicability of the asymptotic results obtained here can be investigated on the basis of a numerical solution of the integral equation (2.5) using the mechanical quadrature method [5].

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